International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

Presenting the Straddle Lemma in an introductory Real Analysis course

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INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY, 2017
VOL. 48, NO. 3, 428–482

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CLASSROOM NOTES

Presenting the Straddle Lemma in an introductory Real Analysis course

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ABSTRACT
In this article, we revisit the concept of strong differentiability of real functions of one variable, underlying the concept of differentiability. Our discussion is guided by the Straddle Lemma, which plays a key role in this context. The proofs of the results presented are designed to meet a young audience in mathematics, typical of students in a first course of Real Analysis or an honors-level Calculus course.

ARTICLE HISTORY
Received 21 July 2016

KEYWORDS
Straddle Lemma; strong derivative; continuously differentiable function; geometric interpretation of the derivative; tangent line

1. Introduction

One of the most powerful illustrations of the concept of the derivative to newcomers to Calculus is the traditional picture of a secant line crossing the graph of a function f at two points P and Q, and how this secant line becomes tangent to the graph when we keep P fixed and slide Q toward P along the graph, as shown in Figure 1.

Conveniently, when trying to compute the slope of the secant line in this picture, the situation translates almost perfectly into the limit that defines the derivative of f at x_0:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{1}$$

Some minor details differ; of course, x may approach x_0 from either side in (1), while in Figure 1 the point Q is almost always pictured to come toward P from the right-hand side – which corresponds to x approaching x_0 with values greater than x_0 – but it is just as easy to imagine a similar picture where Q starts at the left-hand side of P.

A geometrically inclined beginner might ask why not adopt a more symmetrical approach and make points P and Q straddle toward each other, creating a tangent line at whatever point P and Q end up meeting – perhaps even fixing a point T beforehand between P and Q where the desired tangent line should touch the graph. Figure 2 illustrates this approach.

The reason why we do not see this alternate geometric picture discussed in Calculus books becomes clear when we consider the analytical analogue of Figure 2. We need to

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http://dx.doi.org/10.1080/0020739X.2016.1242786

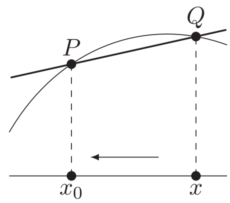

Figure 1. Usual geometric view of a tangent line.

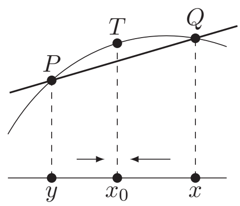

Figure 2. Alternate geometric view of a tangent line.

consider the limit

$$\lim_{x,y \to x_0} \frac{f(x) - f(y)}{x - y}, \tag{2}$$

where now x and y are allowed to approach x_0 independently, subject only to the restriction that we never have $x = y$. There are some complications with this new expression as compared to (1). First, (2) is a limit in two variables, which requires a discussion of this more general kind of limit; second, (2) produces a more restrictive kind of differentiability than (1); a function for which the limit (2) exists is termed *strongly differentiable* – a notion that is related to f being continuously differentiable – thus not quite the basic differentiability that one seeks to explore in a first course.

If, however, following the geometric spirit of Figure 2, we insist that x and y always approach x_0 from *opposite* sides in (2), we do obtain a notion of derivative which is equivalent to the usual definition. This is the statement of the so-called *Straddle Lemma:* if f is differentiable at x_0, then the limit in (2) exists so long as x and y approach x_0 from opposite sides. Despite being well known, discussion and proof of the Straddle Lemma is usually confined to the context of Integration theory, and as such uses language and tools which are out of reach to a beginner (see, for example, [1,2], [3, p. 2], [4, p. 138], [5, p. 670]). We give an elementary proof of the Straddle Lemma which requires only discussion of limits in two variables in addition to the machinery available to a student in a first course of Real Analysis. This proof allows the student to see that the alternate geometric interpretation of the tangent line as given in Figure 2 is indeed equivalent to the more traditional picture.

We further discuss strongly differentiable functions in Section 3 at the same level of the presentation of the Straddle Lemma.

2. Proving the Straddle Lemma

Since the limit (2) is a limit in two variables, we must give a suitable definition for this type of limit in order to be able to discuss the Straddle Lemma in more detail. The following definition is a natural extension of the definition of a limit in one variable.

Definition 2.1: Let D be a subset of $\mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ [we do not require that D contains the point (x_0, y_0), but we do not exclude this possibility]. Assume that for any positive real number s, we have that the 'deleted' square

$$R_s^*(x_0, y_0) = [(x_0 - s, x_0 + s) \times (y_0 - s, y_0 + s)] \setminus \{(x_0, y_0)\}$$

satisfies $D \cap R_s^*(x_0, y_0) \neq \emptyset$. If φ is a real function of two variables defined on D, then we write

$$\lim_{(x,y)\to(x_0,y_0)} \varphi(x, y) = L$$

if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $(x, y) \neq (x_0, y_0)$ in D we have that $|x - x_0| < \delta$ and $|y - y_0| < \delta$ imply

$$|\varphi(x, y) - L| < \varepsilon.$$

Remark 2.1: Now that we have a rigorous definition for the limit of a function in two variables, we agree to retroactively consider (2) as representing such a limit, where the quotient $[f(x) - f(y)]/(x - y)$ is regarded as a function in two variables defined for all (x, y) such that $x \neq y$ and both x and y are in the domain of f; the notation $x, y \to x_0$ below the word 'lim' is to be understood as $(x, y) \to (x_0, x_0)$. In the statement of Straddle Lemma 2.1, we will also use a special notation to indicate a restriction on the domain where we will be taking the limit.

We are now ready to state the Straddle Lemma.

Straddle Lemma 2.1: *Let f be a real function defined on the interval (a, b). If f is differentiable at $x_0 \in (a, b)$, then we must have*

$$\lim_{(x,y)\to(x_0^-,x_0^+)} \frac{f(x) - f(y)}{x - y} = f'(x_0), \tag{3}$$

where the notation $(x, y) \to (x_0^-, x_0^+)$ below the word 'lim' *means we are restricting the quotient in the limit above to points (x, y) such that $x \leq x_0 \leq y$.*

Note that the restriction $x \leq x_0 \leq y$ in the limit (3) means that the limit is taken as x approaches x_0 from the left-hand side and y approaches x_0 from the right-hand side. In addition, the definition of a limit in two variables allows one of x or y to be set equal to x_0, provided that the other is not. This latter fact enables us to show that standard differentiability follows from the existence of the limit on the left-hand side of Equation (3). Thus, we have the following:

Corollary 2.1: *Let f be as in the statement of the Straddle Lemma, and let $x_0 \in (a, b)$ (we do not assume that f is differentiable at x_0). Then f is differentiable at x_0 if and only if*

$$\lim_{(x,y)\to(x_0^-,x_0^+)} \frac{f(x)-f(y)}{x-y} \tag{4}$$

exists.

Proof: The Straddle Lemma shows that if f is differentiable at x_0, then (4) exists. For the other direction, assume (4) exists. Setting $y = x_0$ in (4) and letting x approach x_0 from the left-hand side shows that the left-hand derivative of f exists at x_0; analogously, setting $x = x_0$ in (4) and letting y approach x_0 from the right-hand side establishes the existence of the right-hand derivative of f at x_0. Moreover, since we are assuming (4) exists, it follows that the left-hand side and right-hand side derivatives must both be equal to the value of the limit (4), so they must be equal to each other. Thus, f is differentiable at x_0. ■

Note also that while (3) is a limit in two variables, the proof below shows that it can be reduced essentially to two limits in a single variable.

Proof of Straddle Lemma 2.1: Put

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & (x \neq x_0) \\ f'(x_0) & (x = x_0) \end{cases}, \qquad h(y) = \begin{cases} \frac{f(y)-f(x_0)}{y-x_0} & (y \neq x_0) \\ f'(x_0) & (y = x_0) \end{cases}.$$

To simplify the discussion, let D be the set of all $(x, y) \in \mathbb{R}^2$ such that $x, y \in (a, b)$, $x \leq x_0 \leq y$ and $x \neq y$. We then have for all $(x, y) \in D$:

$$\begin{aligned}\left|\frac{f(x)-f(y)}{x-y} - f'(x_0)\right| &= \left|\frac{x-x_0}{x-y} g(x) + \frac{x_0-y}{x-y} h(y) - \frac{(x-x_0)+(x_0-y)}{x-y} f'(x_0)\right| \\ &\leq \left|\frac{x-x_0}{x-y}\right| \left|g(x)-f'(x_0)\right| + \left|\frac{x_0-y}{x-y}\right| \left|h(y)-f'(x_0)\right|.\end{aligned}$$

Now note that since x and y are on opposite sides of x_0, we have $|x - x_0| \leq |x - y|$ and $|x_0 - y| \leq |x - y|$ (the possibility of equality must be considered because one of x, y may be equal to x_0). This allows us to write

$$\left|\frac{f(x)-f(y)}{x-y} - f'(x_0)\right| \leq \left|g(x)-f'(x_0)\right| + \left|h(y)-f'(x_0)\right|. \tag{5}$$

Now pick $\varepsilon > 0$. Differentiability of f at x_0 implies that there exists $\delta > 0$ such that if $x_0 - \delta < x < x_0 < y < x_0 + \delta$, then

$$\left|g(x)-f'(x_0)\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|h(y)-f'(x_0)\right| < \frac{\varepsilon}{2}.$$

On the other hand, $x = x_0$ implies $g(x) = f'(x_0)$, so in this case we also have $|g(x) - f'(x_0)| < \varepsilon/2$; analogously, $y = x_0$ implies $|h(y) - f'(x_0)| < \varepsilon/2$. This shows that for any $(x, y) \in D$

such that $|x - x_0| < \delta$ and $|y - x_0| < \delta$, the left-hand side of (5) is less than ε, which finishes the proof.

Remark 2.2: We used the functions g, h in the foregoing proof to avoid the difficulty presented by the fact that when one of x, y is equal to x_0, we cannot divide by $x - x_0$ or $y - x_0$ to form the single-variable differential quotients.

The conclusion of the Straddle Lemma fails to hold if we allow the pair (x, y) to approach (x_0, x_0) freely, i.e. if we consider the limit (2) in place of the left-hand side of Equation (3). To see an example, put

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & (x \neq 0) \\ 0 & (x = 0) \end{cases}.$$

Recall that as a consequence of the Squeeze theorem from elementary Calculus, if $k(x)$ is bounded and $\lim_{x\to x_0} l(x) = 0$, then $\lim_{x\to x_0} k(x)l(x) = 0$. This implies that f is differentiable at 0 with $f'(0) = 0$. However, if we let x and y approach 0 via the sequences $x_n = 1/(\pi/2 + n\pi)$, $y_n = x_{n+1}$ we see that the limit (2) does not exist in this case: $[f(x_n) - f(y_n)]/(x_n - y_n)$ has limit $2/\pi$ if we restrict n to even integers and has limit $-2/\pi$ for odd n.

This shows that the two-variable limit (2) defines a special kind of differentiability. We briefly discuss this in the next section.

3. Strongly differentiable and continuously differentiable functions

For completeness, we now consider the situation when the limit (2) exists. For a more thorough and general treatment, see [6–9].

Definition 3.1: A real function f defined on an interval (a, b) is said to be *strongly differentiable* at a point $x_0 \in (a, b)$ if

$$\lim_{(x,y)\to(x_0,x_0)} \frac{f(x) - f(y)}{x - y} \tag{6}$$

exists.

Obviously, setting $y = x_0$ in (6) shows that if f is strongly differentiable at x_0, then it is differentiable in the usual sense at x_0 [and (6) must be equal to $f'(x_0)$]. On the other hand, in light of the foregoing definition, we see that the example that concluded the previous section shows that a function can be differentiable at a point without being strongly differentiable at the same point. Thus, strong differentiability is indeed a stronger condition than ordinary differentiability.

The next theorem relates strong differentiability of a function with the continuity of its derivative.

Theorem 3.1: *Let f be differentiable on the interval (a, b). Then, f is strongly differentiable at $x_0 \in (a, b)$ if and only if f' is continuous at x_0.*

We use an expanded version of the proof in [8, p. 971] adapted to our setting.

Proof: Assume f is strongly differentiable at x_0, and pick $\varepsilon > 0$. Let $\delta > 0$ be such that if distinct x, y satisfy $|x - x_0| < \delta$ and $|y - x_0| < \delta$, then,

$$\left|\frac{f(x) - f(y)}{x - y} - f'(x_0)\right| < \frac{\varepsilon}{2}$$

[we may assume δ is small enough so that (a, b) contains every such x, y – then, we need not worry if $f'(x)$ and $f'(y)$ are defined]. Now fix x such that $|x - x_0| < \delta$. By differentiability of f at x, there is $\delta' > 0$ such that $0 < |y - x| < \delta'$ implies

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \frac{\varepsilon}{2}.$$

Then, if we set $\delta_1 = \min\{\delta', \delta - |x - x_0|\}$, both inequalities are satisfied for all y such that $0 < |y - x| < \delta_1$. Applying the triangle inequality yields that $0 < |y - x| < \delta_1$ implies

$$|f'(x) - f'(x_0)| < \varepsilon,$$

but since this last inequality does not depend on y, we conclude that it holds if $|x - x_0| < \delta$. This establishes one direction of the proof.

For the other direction, assume f' is continuous at x_0 and again pick $\varepsilon > 0$. Let $\delta > 0$ be such that $|\xi - x_0| < \delta$ implies $|f'(\xi) - f'(x_0)| < \varepsilon$. Now take any distinct x, y such that $|x - x_0| < \delta$ and $|y - x_0| < \delta$. By the Mean Value theorem,

$$\frac{f(x) - f(y)}{x - y} = f'(\xi),$$

where ξ sits between x and y. Thus, $|\xi - x_0| < \delta$ and

$$\left|\frac{f(x) - f(y)}{x - y} - f'(x_0)\right| = |f'(\xi) - f'(x_0)| < \varepsilon.$$

This concludes the proof. ■

Theorem 3.1 shows that a function f is strongly differentiable on every point of an interval if and only if f is continuously differentiable on the interval. Note, however, that strong differentiability is a pointwise condition, while on the other hand, it does not make sense to talk about a function being continuously differentiable at a single point. To emphasize this distinction, we conclude with an example of a function which is strongly differentiable at 0 but not even differentiable on any open interval containing 0: define f by $f(0) = 0, f(1/n) = 1/n^2$ for all nonzero integers, and complete the graph of f by connecting linearly the points already defined.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Visualization in mechanics: the dynamics of an unbalanced roller

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ABSTRACT

It is well known that mechanical engineering students often find mechanics a difficult area to grasp. This article describes a system of equations describing the motion of a balanced and an unbalanced roller constrained by a pivot arm. A wide range of dynamics can be simulated with the model. The equations of motion are embedded in a graphical user interface for its numerical solution in MATLAB. This allows a student's focus to be on the influence of different parameters on the system dynamics. The simulation tool can be used as a dynamics demonstrator in a lecture or as an educational tool driven by the imagination of the student. By way of demonstration the simulation tool has been applied to a range of roller–pivot arm configurations. In addition, approximations to the equations of motion are explored and a second-order model is shown to be accurate for a limited range of parameters.

ARTICLE HISTORY
Received 10 August 2016

KEYWORDS
Mechanics; unbalanced systems; approximations to dynamic systems; GUI; animation

1. Introduction

Some mechanical engineering undergraduate students find mechanics a challenging topic. This is a well-recognized problem [1–4]. The reason for this is complex and no one issue is the root cause of this but one element of the problem is mechanics is a highly mathematical topic that requires students to be more than competent in the formation and solution of

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